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Fundamental Frequencies of Elliptical Plates using Static Deflections

Murat ALTEKIN[1]

ABSTRACT

Fundamental frequencies of solid and annular elliptical plates were approximated using the static deflections by means of finite element method (FEM) without computing the eigenvalues. The problem was formulated within the framework of the first order shear deformation theory (FSDT). The effects of (i) the inner and outer boundary conditions, (ii) the size of the perforation, (iii) the aspect ratio, and (iv) the thickness of the plate on the performance of the method were examined via a large number of numerical simulations. Convergence study was performed through h-refinement. Accuracy of the results was validated through comparison studies. The results reveal that the application of Morley's formula which does not require eigenvalue analysis approximates the fundamental frequency with finer mesh compared to the eigenvalue analysis. The method can be considered as a practical technique to approximate the fundamental frequency. However, the boundary conditions have dominant role on the accuracy of the solution particularly when the plate is perforated.

Keywords: Vibration, fundamental frequency, static deflection, plate, finite element.

1. INTRODUCTION

Plates are lightweight members which are extensively used as primary or secondary structures (e.g., slabs, wings of aircrafts, parts of machines, solar panels, power plants, aircrafts, electronic devices, etc.) in a broad variety of industrial fields by various engineering disciplines (e.g., civil, mechanical, aeronautical, ocean, and naval engineering) [1-6]. Due to architectural demands or design purposes such as reduction of weight, efficiency of materials, and ventilation cutouts of various shapes are frequently incorporated in plate-type structures which may be thin or thick [7-8]. Sensors, actuators [9], and mechanical power transmission components [10] are some of the typical applications of annular plates.

Their widespread use gives rise to the publications in which static or dynamic response of plates is studied (e.g. [1-59]). The majority of the investigations on plates involve the

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1 Department of Civil Engineering, Yildiz Technical University, Istanbul, Turkey
altekin@yildiz.edu.tr - https://orcid.org/0000-0003-3532-688X

classical plate theory (CPT) which is a valid model only for thin plates [50], or FSDT [51]. The superiority of FSDT over CPT is that the transverse shear deformation which should be considered for a reliable analysis of moderately thick plates [51], is included in FSDT. Although FSDT has been proved to be sufficiently effective and accurate [52], it requires a shear correction factor which is difficult to determine [50]. Various higher order shear deformation theories (HSDTs) have been developed to overcome this difficulty [50, 53]. Compared to the solution within the framework of FSDT, the accuracy of the analysis using HSDTs has been slightly improved [50], but the equations of motion in HSDTs are more complicated than those in FSDT [50]. The computational cost of HSDTs, and the simplicity of FSDT might have been the primary motivations for the rigorous use of FSDT [54] in the literature.

Vibration characteristics of structures have vital importance for engineers. Due to the increasing demand for the solutions of such problems [8], dynamic analysis has been an active research topic (e.g., [1-7, 9-35]). Since the boundary conditions and the geometry of the plate are two of the leading factors that affect the availability of closed form solutions (e.g., [36]), various numerical methods have frequently been used in the papers on plates with curvilinear boundaries (e.g., [5, 21-35]). Apart from the widely used numerical procedures which require the computation of the eigenvalues such as Ritz, FEM, finite difference method, discrete singular convolution (DSC), and differential quadrature method (e.g., [3, 20, 23-25, 37, 49, 55, 60]), there are several approximate techniques (e.g., Dunkerley's method, Southwell method, and Morley's formula) which are used to estimate the fundamental frequency of structures especially when an exact solution is not available [27], or the time cost of the eigenvalue analysis is not preferable or affordable. Weiss [28] calculated the fundamental frequency of thin solid circular plates by means of Dunkerley's formula. Jaroszewicz et al. [29] applied the Bernstein-Kieropian simplest lower estimators for calculation of basic natural vibration frequencies of variable-thickness circular plates. Recently, Jaroszewicz, and Radziszewski [30] investigated the approximate fundamental frequency of clamped circular plates using Dunkerley's formula. Altekin [31-32] reported the fundamental frequencies of solid and annular circular plates using Morley's formula.

The engineering motivation of the problem is to determine the fundamental frequencies of solid and annular elliptical plates using the static deflections by means of Morley's formula which is based on Rayleigh's method [40], rather than performing a typical eigenvalue analysis which is inherently non-linear [42]. However, since "the accuracy depends largely on how closely the static deflection shape approximates the fundamental mode" [40], the accuracy is improved with increasing number of elements in the solution domain. To the best of the author's knowledge, there have been no published papers on the dynamic analysis of shear deformable elliptical plates using static deflections. FSDT was adopted in the formulation in the current study. The influence of the boundary conditions, the aspect ratio, and the effect of the perforation on the results were discussed. The algorithm was coded by the author in Matlab. The classical FEM solution which was also presented in the paper to highlight the performance of Morley's method, was coded in Julia as well. The accuracy of the results was validated through comparison studies, and admissible accuracy was obtained.

2. FORMULATION

The family of concentric ellipses is defined by [23]

$$(x/a)^2 + (y/b)^2 = 1 - u, \qquad 0 \le u \le u_i \le 1 \tag{1}$$

where *a* and *b* denote the semi-major, and the semi-minor axes, respectively (Fig. 1). *u* is the variable which is used to generate concentric ellipses with the same aspect ratio such that $c = a/b = a_i/b_i$. Here, a_i and b_i stand for the semi-major, and the semi-minor axes of the elliptical perforation, respectively. The size of the cutout is defined by the parameter *α* given by [23]

$$a/a_i = b/b_i = \alpha, \qquad \alpha = \sqrt{1-u_i}\ . \tag{2}$$

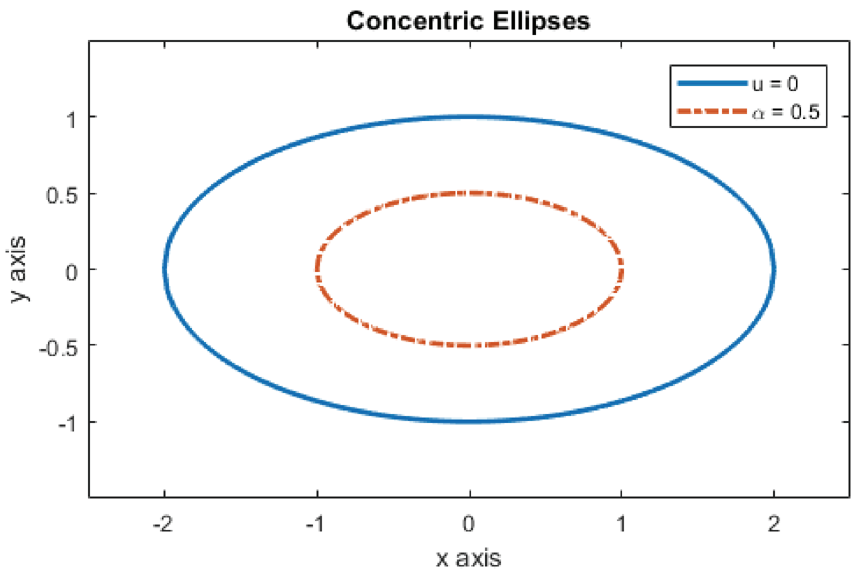

Fig. 1 - Concentric ellipses (c=2)

If $u = 0$ and $u = u_i$ are substituted into Eq. (1) the outer periphery and the inner boundary of the plate are obtained. One-quarter of the plate is regarded as the computational domain in the solution, and the FEM procedure is applied. Four-noded isoparametric quadrilateral plate bending element with straight boundaries [38-39] is used for the discretization of the plate. Configuration of the finite elements for a perforated elliptical plate is shown in Figure 2. The number of elements, and the number of nodes are introduced by *m*, and *n*, respectively. Both *m*, and *n* are determined by the integer *p* which controls the number of partitions such that *m*, and *n* increase with increasing *p*. The geometry of the element and the shape function are identified by [39]:

$$x = \sum_{j=1}^{4} N_j x_j, \qquad y = \sum_{j=1}^{4} N_j y_j, \qquad N_j = \frac{1}{4}\left(1 + r r_j\right)\left(1 + s s_j\right) \tag{3}$$

where r_j and s_j denote the local coordinates r and s of node j [39].

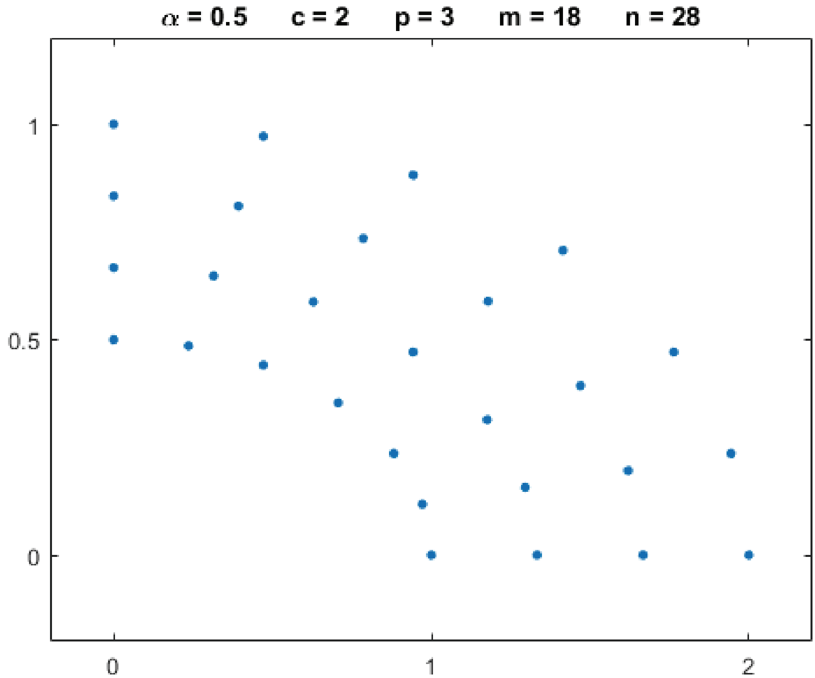

Fig. 2 - Location of the nodes in the first quadrant of a perforated elliptical plate (c=2)

Each element has three degrees of freedom per node defined by [39]

$$w = \sum_{j=1}^{4} N_j w_j, \qquad \theta_x = \sum_{j=1}^{4} N_j \theta_{xj}, \qquad \theta_y = \sum_{j=1}^{4} N_j \theta_{yj}\,. \tag{4}$$

Here, the normal deflection is indicated by *w*, and the rotations are denoted by θ_x and θ_y. The relation between the curvature and shear deformation vector ***{ε}*** and the nodal displacement vector ***{d_j}*** is given by [39]

$$\{d_j\}^T = \{w_j \quad \theta_{xj} \quad \theta_{yj}\}, \qquad \{\varepsilon\} = \sum_{j=1}^{4} [B_j]\{d_j\}, \tag{5}$$

$$\{\varepsilon\} = \begin{Bmatrix} k_x = \sum_{j=1}^{4} \theta_{yj} \dfrac{\partial N_j}{\partial x} \\ k_y = -\sum_{j=1}^{4} \theta_{xj} \dfrac{\partial N_j}{\partial y} \\ k_{xy} = \sum_{j=1}^{4} \theta_{yj} \dfrac{\partial N_j}{\partial y} - \sum_{j=1}^{4} \theta_{xj} \dfrac{\partial N_j}{\partial x} \\ \phi_x = \sum_{j=1}^{4} w_j \dfrac{\partial N_j}{\partial x} + \sum_{j=1}^{4} \theta_{yj} N_j \\ \phi_y = \sum_{j=1}^{4} w_j \dfrac{\partial N_j}{\partial y} - \sum_{j=1}^{4} \theta_{xj} N_j \end{Bmatrix}, \tag{6}$$

$$[B_j] = \begin{bmatrix} 0 & 0 & 0 & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_j}{\partial y} \\ 0 & -\dfrac{\partial N_j}{\partial y} & -\dfrac{\partial N_j}{\partial x} & 0 & -N_j \\ \dfrac{\partial N_j}{\partial x} & 0 & \dfrac{\partial N_j}{\partial y} & N_j & 0 \end{bmatrix}^T . \tag{7}$$

The element stiffness matrix is given by

$$[k_e] = \iint_A [B]^T [C][B]\,dxdy, \qquad [k_e] = [k_B] + [k_S] \tag{8}$$

where [39]

$$[C] = \begin{bmatrix} [C_B] & [0] \\ [0] & [C_S] \end{bmatrix}, \qquad [C_B] = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{(1-\nu)}{2} \end{bmatrix}, \tag{9}$$

$$[B]=\left[[B_1]\ \ [B_2]\ \ [B_3]\ \ [B_4]\right], \qquad D=\frac{Eh^3}{12\left(1-\nu^2\right)}, \qquad G=\frac{E}{2\left(1+\nu\right)} \tag{10}$$

$$\left[C_S\right]=D_S\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad D_s = Gh\kappa\,. \tag{11}$$

Here, h is the thickness of the plate, D denotes the bending rigidity, and κ is the shear correction factor. Selective integration is performed to prevent shear locking [38-39]. The nodal displacements are determined by

$$[K]\{U\}=\{F\}\,. \tag{12}$$

3. ANALYSIS

The plate is divided into m sections ("sections" refer to "elements"), and it is assumed that the mass of each section (element) is lumped at discrete points [40]. Morley's formula (or Rayleigh quotient) is given by

$$\omega^2 = g\sum_{i=1}^{m} q_i w_i \Big/ \sum_{i=1}^{m} q_i w_i^{\,2} \tag{13}$$

where q_i is the weight of the ith element, and ω denotes the natural circular frequency [40]. In the current paper the thickness of the homogeneous and isotropic plate is uniform. It is assumed in the solution that the plate is subjected to uniformly distributed transverse pressure. Such an external load represents the weight of the plate. First, the nodal displacements of the plate are determined. Next, the area of each quadrilateral element is found. Then, the centroidal deflection of each quadrilateral region is computed via interpolation. Finally, Morley's formula is employed, and the fundamental frequency is obtained.

4. NUMERICAL SIMULATION

Due to symmetry the first quadrant of the plate is regarded as the computational domain, and it is divided into quadrilateral elements (Fig. 2). Numerical simulations were made to investigate the effects of several parameters on the fundamental frequency. Annular elliptical plates were identified by two letter symbols [41]. The first and the second letters relate to the inner edge and the outer edge, respectively [41]. For instance, F-C indicates an annular plate with a free (F) inner edge, and a clamped (C) outer edge [41]. Likewise, C-S denotes an annular plate with a clamped (C) inner edge, and a simply supported (S) outer edge [41]. Convergence study for h-refinement was presented for several values of p which controls the number of nodes in the quarter of the plate (Table 1). For convenience, the non-dimensional frequency parameter (λ), and the parameter of thickness (η) defined by

$$\lambda = \omega a^2 \sqrt{\frac{m_p}{D}}, \qquad \eta = \frac{h}{b} \tag{14}$$

were used. Here, m_p stands for the mass of the plate per unit area. Unless otherwise stated, $\nu=0.3$, and $\kappa=5/6$ were used in the solution.

5. NUMERICAL RESULTS

The classical FEM solution that involves the computation of the eigenvalues was also included in the study to test the performance of Morley's method. The accuracy of the results was validated through comparison studies presented for particular cases (Tables 1-3). Some of the solutions of annular elliptical plates for a large variety of plate categories ranging from thin to moderately thick were presented in Tables 4-9, and the other solutions were presented in Appendix A (Tables A1-A10). The ratio $\lambda_{66} / \lambda_{36}$ was used to examine the performance of Morley's formula where λ_{66} and λ_{36} denote the non-dimensional fundamental frequencies obtained using statics deflections (for $p=66$), and FEM (for $p=36$), respectively (Figs. 3-6).

- Compared to solid plates, annular plates require more refined meshes to obtain the same accuracy. This statement holds for both methods. Admissible accuracy can easily be achieved for solid plates.
- The performance of Morley's formula depends highly on the boundary conditions. Even for solid plates, the solution of (S) plates is more accurate than that of (C) plates. (Table 1).
- The accuracy of Morley's formula increases with decreasing size of the cutout (Tables 2-9). The performance of Morley's formula is very sensitive to the boundary conditions. The highest accuracy is achieved for F-S, and S-F plates.
- The performance of Morley's formula decreases with increasing aspect ratio (Figs. 3-6). The error is about only 1% for circular plates.
- Combined effects of the thickness and the size of the perforation on the results depend on the aspect ratio.
- Compared to FEM solution, the use of Morley's formula requires finer mesh, and consequently, it needs more computer storage requirement since larger matrix operations are performed. On the other hand, the execution time of the code spent for the computation of the eigenvalues is much longer. Therefore, FEM provides less memory storage with longer runtime, whereas Morley's formula requires more memory storage with shorter runtime of the code.
- The convergence study reveals that although fine mesh was used in the study, the rate of convergence of the solution obtained by means of Morley's formula is slow compared to eigenvalue analysis by means of FEM. Good agreement was obtained in the comparison studies for $p=36$, and $p=66$ for FEM, and Morley's method, respectively. The accuracy of the results can be improved with increasing p.

Table 1 - Convergence and comparison of λ for a solid elliptical plate ($\kappa = \pi^2/12$)*

		c=1	c=1	c=2	c=2
Reference	η	(C)	(S)	(C)	(S)
[35]	thin	10.216	4.9351	27.377	13.213
[5]	thin	10.216		27.377	
p=66	0.002	10.32856	4.947658	28.04912	13.30464
p=36 (FEM)	0.002	10.21788	4.935512	27.38297	13.21493
[22]	0.100	9.931			
p=66	0.100	10.09877	4.919172	27.55031	13.213
p=64	0.100	10.09881	4.919191	27.55042	13.21305
p=62	0.100	10.09886	4.919211	27.55053	13.2131
p=36 (FEM)	0.100	9.945924	4.895016	26.81448	13.10122
[34]*	0.100	9.941	4.894		
p=66*	0.100	10.0958	4.9188	27.5439	13.2120
p=36 (FEM)*	0.100	9.9428	4.8946	26.8078	13.1002

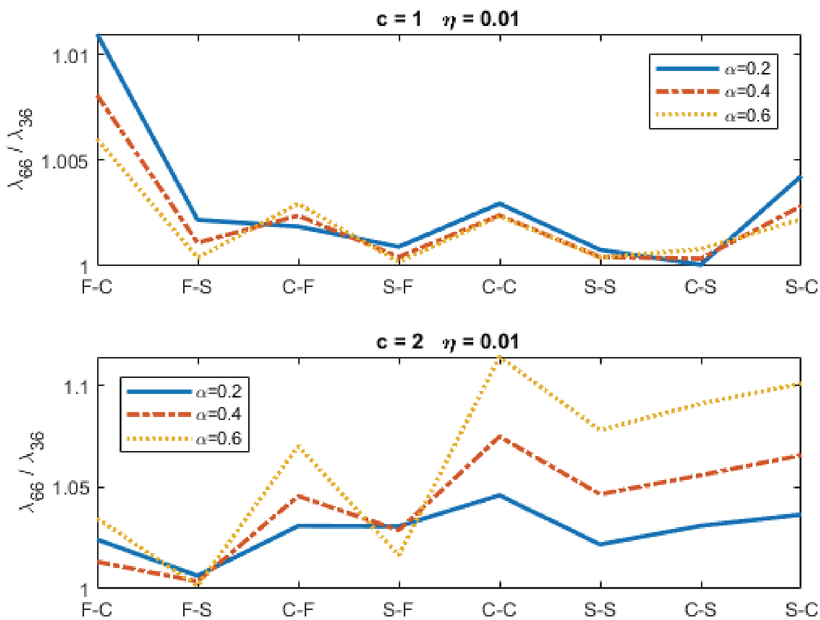

Fig. 3 - Performance of Morley's formula (η=0.01)

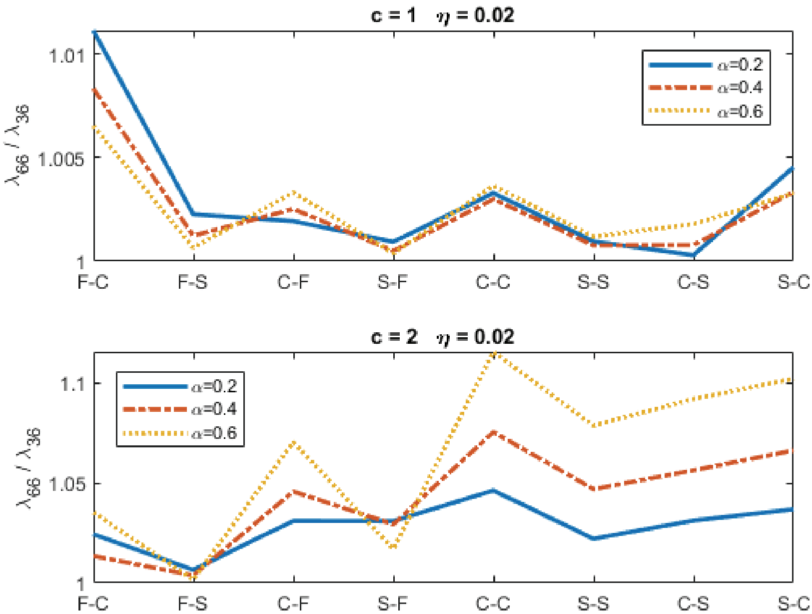

Fig. 4 - Performance of Morley's formula (η=0.02)

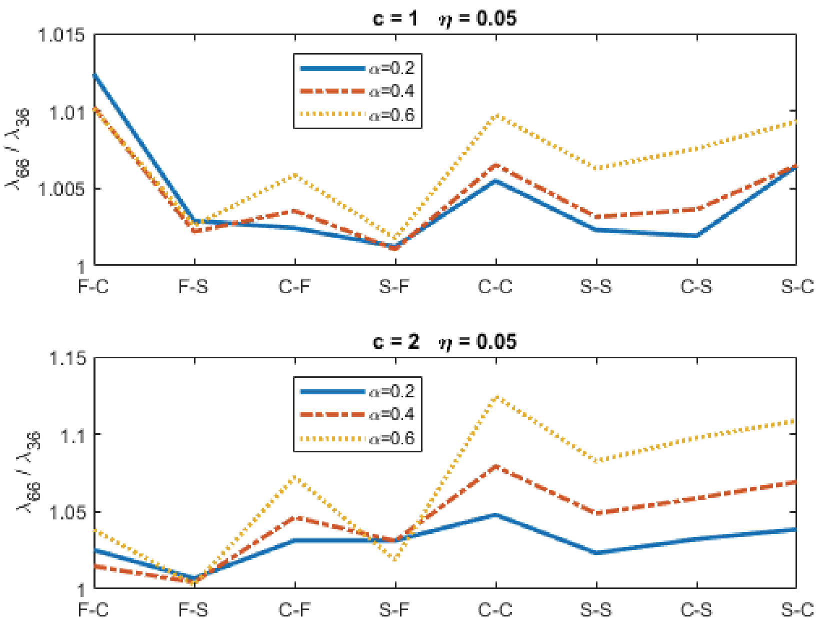

Fig. 5 - Performance of Morley's formula (η=0.05)

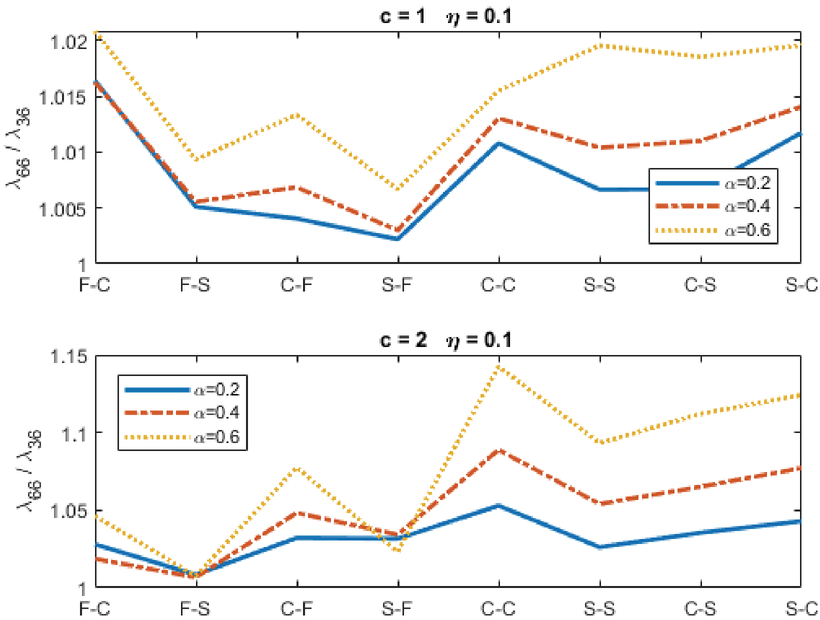

Fig. 6 - Performance of Morley's formula (η=0.10)

Table 2 - Comparison of λ for an annular circular plate
$(\kappa = \pi^2/12,\ \nu = 0.3,\ \alpha = 0.5,\ c = 1)$

Reference	η	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
[2]	0.001	17.714	5.0769	13.024	4.1210	89.248	40.043	59.819	63.972
p=66	0.001	17.83635	5.08058	13.06239	4.122594	89.65655	40.09287	59.94969	64.21778
p=36 (FEM)	0.001	17.71669	5.077479	13.0289	4.12163	89.47581	40.08485	59.93064	64.07627
[20]	0.010					89.028			
p=66	0.010	17.83115	5.080441	13.05857	4.122462	89.40652	40.06908	59.85334	64.12041
p=36 (FEM)	0.010	17.7094	5.07709	13.0241	4.121309	89.20003	40.05427	59.8213	63.96385
[2]	0.050	17.533	5.0655	12.905	4.1135	83.051	39.297	57.250	61.327
p=66	0.050	17.70612	5.077075	12.96667	4.119285	83.88555	39.50478	57.64512	61.87609
p=36 (FEM)	0.050	17.53489	5.066203	12.90945	4.114152	83.23865	39.33776	57.34992	61.4204
[2]	0.100	17.024	5.0321	12.568	4.0915	70.277	37.326	51.219	55.090
[20]	0.100					73.192			
p=66	0.100	17.32577	5.066598	12.68914	4.1094	71.41084	37.88385	52.02557	56.06585
p=36 (FEM)	0.100	17.02585	5.032749	12.57189	4.092101	70.39852	37.36243	51.29502	55.16269

Table 3 - Comparison of λ for an annular thin elliptical plate ($\kappa = 5/6,\ \nu = 1/3,\ \eta = 0.010$)

Reference	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
[3]	1	0.2	10.46	4.851	5.384	3.466	35.12	16.86	23.34	26.68
p=66	1	0.2	10.46079	4.744199	5.225187	3.316449	34.7595	16.75183	22.79789	26.75026
p=36 (FEM)	1	0.2	10.34657	4.733602	5.215471	3.313606	34.65731	16.73882	22.79646	26.6368
[3]	1	0.4	13.50	4.748	9.082	3.634	61.88	28.08	41.27	44.93
p=66	1	0.4	13.60651	4.749357	9.094636	3.631488	62.04021	28.11009	41.29792	45.07354
p=36 (FEM)	1	0.4	13.49753	4.744096	9.073019	3.630024	61.89147	28.0981	41.28357	44.94572
[17]	1	0.5	17.51	5.04	13.05	4.06	89.30	40.01	59.91	64.06
[23]	1	0.5	17.6	5.051			89.25	40.01	59.91	63.85
p=66	1	0.5	17.71343	5.046765	13.12382	4.071506	89.39738	40.03639	59.92877	63.99972
p=36 (FEM)	1	0.5	17.59267	5.043425	13.08911	4.070423	89.19059	40.02144	59.89641	63.84337
[3]	1	0.6	25.24	5.663	20.60	4.809	139.6	62.12	94.26	98.79
p=66	1	0.6	25.68104	5.665307	20.66217	4.809878	139.6056	62.14223	94.23564	98.8963
p=36 (FEM)	1	0.6	25.52804	5.663133	20.60139	4.809027	139.276	62.11734	94.16034	98.67716
[3]	2	0.2	28.03	12.79	8.167	5.895	66.09	35.74	43.46	56.22
p=66	2	0.2	28.25411	12.40696	6.411882	5.281222	61.43884	33.41811	38.52188	54.84044
p=36 (FEM)	2	0.2	27.57337	12.32522	6.212873	5.117726	58.73469	32.71354	37.35849	52.92731
[3]	2	0.4	36.35	12.26	12.19	7.666	94.54	49.93	62.90	78.41
[56]	2	0.4	36.428							
p=66	2	0.4	36.71034	12.23611	11.57788	7.04996	97.51724	50.60931	62.75333	81.80067
p=36 (FEM)	2	0.4	36.22175	12.19083	11.0594	6.846701	90.71251	48.36802	59.41423	76.77172
[23]	2	0.5	45.76	13.06			128.9	66.21	87.84	102.6
p=66	2	0.5	46.1018	12.97565	16.79966	8.485305	133.2842	67.50973	86.78968	108.3289
p=36 (FEM)	2	0.5	45.42884	12.94443	15.8843	8.286729	121.9813	63.63252	80.9714	100.1394
[3]	2	0.6	59.75	14.56	25.38	10.84	181.7	91.78	122.5	142.7
p=66	2	0.6	61.67075	14.55428	26.41716	10.61525	198.614	97.87311	130.7351	156.0929
p=36 (FEM)	2	0.6	59.54442	14.533	24.64509	10.44007	178.2366	90.79605	119.776	141.798

Table 4 - Nondimensional frequency parameter λ for an annular circular plate
($m = 8712,\ \kappa = 5/6,\ \nu = 0.3$)

	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=66	0.01	1	0.1	10.27533	4.866091	4.246569	3.454977	27.42665	14.50681	17.81764	22.83659
p=66	0.01	1	0.2	10.52178	4.728858	5.19275	3.342947	34.76155	16.80173	22.74546	26.86897
p=66	0.01	1	0.3	11.53152	4.672342	6.676209	3.424803	45.51021	21.10243	30.01327	33.90118
p=66	0.01	1	0.4	13.70983	4.769607	9.043253	3.67466	62.04628	28.1477	41.2328	45.18836
p=66	0.01	1	0.5	17.83122	5.080443	13.05862	4.122464	89.4098	40.06939	59.85461	64.12169
p=66	0.01	1	0.6	25.8166	5.713021	20.57742	4.871151	139.6356	62.17265	94.15094	99.03704

Table 5 - Nondimensional frequency parameter λ for an annular circular plate
$(m = 8712, \quad \kappa = 5/6, \quad \nu = 0.3)$

	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=66	0.02	1	0.1	10.26871	4.8653	4.244113	3.453594	27.34163	14.48966	17.78078	22.79219
p=66	0.02	1	0.2	10.51551	4.728204	5.190017	3.342171	34.63947	16.78556	22.69566	26.81612
p=66	0.02	1	0.3	11.52458	4.671804	6.672452	3.424258	45.31072	21.08042	29.93423	33.81993
p=66	0.02	1	0.4	13.70047	4.769148	9.037208	3.67422	61.68519	28.11128	41.09183	45.04503
p=66	0.02	1	0.5	17.81571	5.080028	13.0472	4.122072	88.67334	39.99846	59.56906	63.83279
p=66	0.02	1	0.6	25.78418	5.712617	20.55115	4.870768	137.8675	62.00594	93.46592	98.34604

Table 6 - Nondimensional frequency parameter λ for an annular circular plate
$(m = 8712, \quad \kappa = 5/6, \quad \nu = 0.3)$

	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=66	0.05	1	0.1	10.22261	4.859776	4.227019	3.443957	26.76735	14.3714	17.52907	22.4881
p=66	0.05	1	0.2	10.4718	4.723629	5.170982	3.336756	33.81776	16.67371	22.35586	26.45389
p=66	0.05	1	0.3	11.47614	4.668049	6.646296	3.420449	43.97994	20.92828	29.39734	33.26545
p=66	0.05	1	0.4	13.63523	4.76594	8.995159	3.671147	59.31244	27.86023	40.14239	44.07494
p=66	0.05	1	0.5	17.70781	5.077121	12.9679	4.119328	83.9542	39.51228	57.67356	61.90514
p=66	0.05	1	0.6	25.55933	5.709787	20.36939	4.868088	127.0362	60.87526	89.03812	93.85425

Table 7 - Nondimensional frequency parameter λ for an annular circular plate
$(m = 8712, \quad \kappa = 5/6, \quad \nu = 0.3)$

	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=66	0.1	1	0.1	10.06173	4.840197	4.167365	3.41014	24.97495	13.97325	16.71164	21.48976
p=66	0.1	1	0.2	10.31862	4.707396	5.10441	3.317617	31.28547	16.29225	21.25577	25.26151
p=66	0.1	1	0.3	11.3062	4.654707	6.554914	3.406945	40.0056	20.41088	27.68858	31.47084
p=66	0.1	1	0.4	13.40679	4.754535	8.848778	3.660232	52.57634	27.01571	37.21098	41.02994
p=66	0.1	1	0.5	17.33224	5.066779	12.69384	4.109572	71.58349	37.91035	52.10979	56.1539
p=66	0.1	1	0.6	24.78697	5.699714	19.75	4.858548	102.0849	57.28992	77.12697	81.57558

Table 8 - Nondimensional frequency parameter λ for an annular elliptical plate
$(m = 8712, \quad \kappa = 5/6, \quad \nu = 0.3)$

	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=66	0.01	2	0.1	27.76433	12.97897	5.195505	4.838216	51.60946	29.15151	31.82645	48.07377
p=66	0.01	2	0.2	28.38664	12.44127	6.429249	5.318576	61.44063	33.4344	38.39179	54.99793
p=66	0.01	2	0.3	31.21666	12.16949	8.434909	6.070422	75.73697	40.22054	47.95923	65.68264
p=66	0.01	2	0.4	36.87373	12.35509	11.57985	7.116293	97.52059	50.62572	62.54868	82.03108
p=66	0.01	2	0.5	46.30899	13.12671	16.79345	8.575032	133.2898	67.52182	86.52295	108.6167
p=66	0.01	2	0.6	62.00036	14.74158	26.39735	10.73895	198.6252	97.87762	130.3741	156.4694

Table 9 - Nondimensional frequency parameter λ of an annular elliptical plate
$(m = 8712,\ \ \kappa = 5/6,\ \ \nu = 0.3)$

	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=66	0.02	2	0.1	27.74381	12.97043	5.190193	4.823988	51.52595	29.0839	31.7808	47.95688
p=66	0.02	2	0.2	28.36601	12.43045	6.423655	5.30274	61.34398	33.37031	38.33973	54.88079
p=66	0.02	2	0.3	31.19722	12.15688	8.428647	6.055412	75.60862	40.14524	47.89198	65.54309
p=66	0.02	2	0.4	36.85242	12.34013	11.57174	7.10352	97.32946	50.52876	62.45162	81.84741
p=66	0.02	2	0.5	46.27514	13.10799	16.78094	8.564884	132.9654	67.38578	86.36362	108.3474
p=66	0.02	2	0.6	61.93203	14.71623	26.3741	10.73086	197.9662	97.6612	130.0627	156.0095

6. CONCLUSIONS

The fundamental frequencies of solid and perforated elliptical plates were computed using static deflections by means of Morley's formula. FEM was implemented to determine the static deflections of the plate. FSDT was adopted in the formulation of the problem. The solution was validated through comparison studies.

The computational results reveal that the main advantage of Morley's formula is that it is a straightforward and simple solution technique for approximating the natural frequencies without performing eigenvalue analysis. However, the deformed form of the plate should be as close as possible to the fundamental mode shape in order to obtain results with admissible accuracy. Owing to this condition, inevitably, fine mesh is required in the analysis. The rate of convergence depends closely on the boundary conditions. It can be concluded that Morley's formula can be used to predict the approximate fundamental frequencies of solid and annular elliptical plates.

Nomenclature

g, h, p : gravitational acceleration, thickness of the plate, number of partitions in the quarter of the plate

r, q, w : radial coordinate, uniform transverse pressure, deflection

D, E, G : flexural rigidity, Young's modulus, shear modulus

k_x , k_y, k_{xy} : curvatures, twist

r_j, s_j : local coordinates of the node j (j=1, 2, 3, 4)

D_s, N_j : shear rigidity, shape function (j=1, 2, 3, 4)

m_p, q_i : mass of the plate per unit area, weight of the i^{th} element

$[k_e]$, $[K]$: element, and global stiffness matrices

[B], [C] : strain-displacement matrix, constitutive matrix

{F}, {U} : global nodal load vector, global displacement vector

$[k_B]$, $[k_S]$: bending, and shear stiffness part of $[k_e]$

$[C_B]$, $[C_S]$	: bending, and shear deformation part of [C]
λ, θ	: nondimensional frequency parameter, transverse coordinate
ω	: natural circular frequency
κ, η, ν	: shear correction factor, parameter of thickness, Poisson's ratio
θ_x, θ_y	: rotations
ϕ_x, ϕ_y	: average shear deformations
$\{\varepsilon\}$, $\{d_j\}$	: curvature and shear deformation vector, nodal displacement vector

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Appendix A Fundamental frequencies of elliptical plates

Table A1 - Nondimensional frequency parameter λ of an annular elliptical plate
($\kappa = 5/6,\ \ \nu = 0.3$)

	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=66	0.05	2	0.1	27.62733	12.9389	5.163452	4.772515	51.00799	28.79606	31.53625	47.39444
p=66	0.05	2	0.2	28.25236	12.39328	6.391149	5.240666	60.70895	33.09019	38.04565	54.28566
p=66	0.05	2	0.3	31.08028	12.11513	8.390859	5.995446	74.7496	39.8203	47.50148	64.82146
p=66	0.05	2	0.4	36.7095	12.29159	11.52259	7.051332	96.03922	50.10879	61.87333	80.87554
p=66	0.05	2	0.5	46.0534	13.04783	16.70599	8.521915	130.7639	66.78147	85.38971	106.8744
p=66	0.05	2	0.6	61.52715	14.63499	26.23721	10.69514	193.4737	96.65242	128.1097	153.3746

Table A2 - Nondimensional frequency parameter λ of an annular elliptical plate
($\kappa = 5/6,\ \ \nu = 0.3$)

	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=66	0.1	2	0.1	27.25841	12.86729	5.107701	4.689845	49.47486	28.18328	30.88405	45.99706
p=66	0.1	2	0.2	27.89727	12.31621	6.304909	5.117169	58.71751	32.42768	37.1991	52.66475
p=66	0.1	2	0.3	30.69696	12.03293	8.281958	5.864725	71.99508	39.02555	46.33504	62.78054
p=66	0.1	2	0.4	36.21748	12.1989	11.37634	6.930247	91.8612	49.05637	60.10567	78.05461
p=66	0.1	2	0.5	45.30089	12.93473	16.48031	8.416539	123.6177	65.22066	82.36526	102.4925
p=66	0.1	2	0.6	60.2246	14.4827	25.82327	10.6032	179.0274	93.9424	121.993	145.3354

Table A3 - Nondimensional frequency parameter λ of an annular circular plate
(FEM solution, $\kappa = 5/6,\ \ \nu = 0.3$)

FEM	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=36	0.01	1	0.1	10.16095	4.854815	4.239875	3.450487	27.33339	14.49218	17.82	22.71585
p=36	0.01	1	0.2	10.40764	4.718579	5.183075	3.339924	34.65944	16.78894	22.74426	26.75526
p=36	0.01	1	0.3	11.42226	4.664591	6.662063	3.422662	45.39191	21.09087	30.00767	33.78549
p=36	0.01	1	0.4	13.60002	4.764384	9.021792	3.673099	61.89774	28.13587	41.21872	45.06032
p=36	0.01	1	0.5	17.70948	5.077093	13.02415	4.12131	89.20341	40.05458	59.82259	63.96517
p=36	0.01	1	0.6	25.66254	5.710824	20.51702	4.870248	139.3069	62.14791	94.07616	98.81788

Table A4 - Nondimensional frequency parameter λ of an annular circular plate
(FEM solution, $\kappa = 5/6,\ \ \nu = 0.3$)

FEM	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=36	0.02	1	0.1	10.15282	4.853632	4.237204	3.448992	27.24102	14.47278	17.77977	22.66629
p=36	0.02	1	0.2	10.39956	4.717498	5.179976	3.339014	34.52571	16.76948	22.68892	26.69485
p=36	0.02	1	0.3	11.41289	4.663549	6.657637	3.421929	45.1727	21.06343	29.91894	33.69194
p=36	0.02	1	0.4	13.587	4.763271	9.014418	3.672367	61.50055	28.08955	41.05951	44.89502
p=36	0.02	1	0.5	17.68755	5.075732	13.00979	4.120418	88.39363	39.96336	59.49884	63.63227
p=36	0.02	1	0.6	25.6165	5.708846	20.48311	4.868893	137.3667	61.93224	93.29816	98.02369

Table A5 - Nondimensional frequency parameter λ of an annular circular plate (FEM solution, $\kappa = 5/6,\ \nu = 0.3$*)*

FEM	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=36	0.05	1	0.1	10.09647	4.845381	4.218652	3.438585	26.6209	14.33938	17.50587	22.32891
p=36	0.05	1	0.2	10.3436	4.709959	5.158445	3.332665	33.63307	16.63539	22.31302	26.28409
p=36	0.05	1	0.3	11.34805	4.656282	6.626918	3.416807	43.72638	20.87481	29.32022	33.05996
p=36	0.05	1	0.4	13.49705	4.75551	8.963356	3.66726	58.92849	27.77279	39.99698	43.79075
p=36	0.05	1	0.5	17.53672	5.06625	12.91073	4.114196	83.30723	39.34522	57.37823	61.44953
p=36	0.05	1	0.6	25.30217	5.695071	20.251	4.859451	125.8104	60.49522	88.36972	92.98754

Table A6 - Nondimensional frequency parameter λ of an annular circular plate (FEM solution, $\kappa = 5/6,\ \nu = 0.3$*)*

FEM	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=36	0.1	1	0.1	9.903308	4.81635	4.154329	3.402203	24.72229	13.89426	16.62559	21.24375
p=36	0.1	1	0.2	10.15168	4.683416	5.083779	3.310321	30.95082	16.18475	21.11429	24.96806
p=36	0.1	1	0.3	11.12634	4.630692	6.520838	3.398744	39.54025	20.24643	27.45337	31.08072
p=36	0.1	1	0.4	13.19183	4.728196	8.788512	3.649243	51.89957	26.73705	36.80524	40.46014
p=36	0.1	1	0.5	17.03251	5.032932	12.57659	4.092272	70.56465	37.38758	51.37575	55.24679
p=36	0.1	1	0.6	24.28046	5.646847	19.48954	4.826287	100.5254	56.19086	75.72119	80.00907

Table A7 - Nondimensional frequency parameter λ of an annular elliptical plate (FEM solution, $\kappa = 5/6,\ \nu = 0.3$*)*

FEM	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=36	0.01	2	0.1	27.07475	12.89165	5.061909	4.705916	49.9703	28.74601	31.19623	46.89436
p=36	0.01	2	0.2	27.71352	12.36115	6.2358	5.159577	58.73685	32.71842	37.23537	53.0603
p=36	0.01	2	0.3	30.62607	12.10697	8.130193	5.885239	71.46238	38.92184	45.98906	62.5087
p=36	0.01	2	0.4	36.38827	12.31036	11.07393	6.916658	90.71647	48.37151	59.22958	76.964
p=36	0.01	2	0.5	45.64588	13.09593	15.89885	8.378603	121.9879	63.63077	80.73704	100.3753
p=36	0.01	2	0.6	59.91472	14.72081	24.66326	10.56417	178.25	90.78511	119.468	142.1002

Table A8 - Nondimensional frequency parameter λ of an annular elliptical plate (FEM solution, $\kappa = 5/6,\ \nu = 0.3$*)*

FEM	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=36	0.02	2	0.1	27.05248	12.8827	5.056732	4.692196	49.88217	28.6755	31.14844	46.77119
p=36	0.02	2	0.2	27.69017	12.34974	6.230057	5.143487	58.62993	32.64789	37.17866	52.92971
p=36	0.02	2	0.3	30.60245	12.09353	8.123538	5.86876	71.31582	38.83712	45.91398	62.35069
p=36	0.02	2	0.4	36.36089	12.29423	11.06497	6.901032	90.49134	48.26135	59.11815	76.75303
p=36	0.02	2	0.5	45.59931	13.07545	15.8844	8.364583	121.5942	63.4735	80.5481	100.0575
p=36	0.02	2	0.6	59.81491	14.69253	24.63489	10.55184	177.4297	90.52549	119.0857	141.5359

Table A9 - Nondimensional frequency parameter λ of an annular elliptical plate (FEM solution, $\kappa = 5/6,\ \ \nu = 0.3$*)*

FEM	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=36	0.05	2	0.1	26.92087	12.84704	5.030552	4.643395	49.33088	28.3771	30.89008	46.18035
p=36	0.05	2	0.2	27.55669	12.30781	6.197077	5.081753	57.92898	32.33943	36.85662	52.27125
p=36	0.05	2	0.3	30.45735	12.04584	8.083991	5.804767	70.3419	38.4682	45.47485	61.53595
p=36	0.05	2	0.4	36.17673	12.23775	11.01151	6.839482	88.99195	47.77492	58.45026	75.63519
p=36	0.05	2	0.5	45.30272	13.00367	15.79882	8.307599	118.9842	62.75531	79.39263	98.32019
p=36	0.05	2	0.6	59.25524	14.59195	24.46842	10.49861	172.0657	89.27894	116.7155	138.342

Table A10 - Nondimensional frequency parameter λ of an annular elliptical plate (FEM solution, $\kappa = 5/6,\ \ \nu = 0.3$*)*

FEM	η	c	α	F-C	F-S	C-F	S-F	C-C	S-S	C-S	S-C
p=36	0.1	2	0.1	26.50014	12.75995	4.973981	4.564428	47.69939	27.72737	30.1902	44.69753
p=36	0.1	2	0.2	27.13612	12.21358	6.108853	4.96051	55.76555	31.60222	35.92572	50.49555
p=36	0.1	2	0.3	29.98335	11.94283	7.969649	5.668513	67.3097	37.55621	44.16602	59.26773
p=36	0.1	2	0.4	35.55259	12.11762	10.85214	6.700999	84.36307	46.53941	56.43275	72.46566
p=36	0.1	2	0.5	44.32941	12.85067	15.54084	8.171946	111.121	60.87652	75.90456	93.35418
p=36	0.1	2	0.6	57.55466	14.37337	23.96595	10.36267	156.6773	85.92528	109.7005	129.2727